

THE IMAGE OF THE STABLE  $J$ -HOMOMORPHISM

DONALD M. DAVIS† and MARK MAHOWALD†

(Received in revised form 9 February 1988)

## 1. INTRODUCTION

IN THIS PAPER we give detailed proofs of results about the image of the stable  $J$ -homomorphism similar to some announced in [24, 25, 26, and 27]. The methods are those introduced in those papers; we just take a bit more care here to clarify certain aspects of the proof. The paper could be viewed as a response to Frank Adams' challenge ([1]): "for the classical  $J$ -homomorphism, we have no published proof independent of the ideas I am now discussing (the Adams conjecture); homotopy theorists know in principle where they should look for an independent proof, but nobody has yet been willing to undertake the heavy task of working it out in detail and writing it down properly."

The  $J$ -homomorphism, as introduced by G. W. Whitehead in [41], is the homomorphism

$$\pi_i(SO(n)) \xrightarrow{\Sigma^n} \pi_{i+n}(\Sigma^n SO(n)) \xrightarrow{\mu^*} \pi_{i+n}(S^n),$$

where  $\mu$  is defined by applying the Hopf construction to the map

$$S^{n-1} \times SO(n) \rightarrow S^{n-1},$$

which is the action of the special orthogonal group  $SO(n)$  (which we will also write as  $SO_n$ ) on  $S^{n-1} \subset \mathbb{R}^n$ . These are compatible, with the proper sign convention, as  $n$  increases, and for all  $n > i + 1$  they are equal, defining the stable  $J$ -homomorphism

$$J: \pi_i(SO) \rightarrow \pi_i(S^0),$$

where  $\pi_i(S^0)$  denotes the  $i$ th stable homotopy group of spheres ( $i$ th stable stem). By Bott periodicity  $\pi_i(SO)$  is infinite cyclic for  $i \equiv 3 \pmod{4}$ .

We will focus our attention on the image of the 2-primary stable  $J$ -homomorphism in stems  $i \equiv 7 \pmod{8}$ , which is by far the most difficult case. The analogues at odd primes will be discussed in another paper. Let  $v(n)$  denote the exponent of 2 in  $n$ , and  $d(n)$  denote the grading of the  $n$ th nonzero homotopy group of  $SO$ . Thus if  $0 \leq b \leq 3$ , then  $d(4a + b) = 8a + 2^b - 1$ . This is closely related to the so-called vector-fields numbers. We use "d" to suggest "double". Our main result is

**THEOREM 1.1.** *Let  $j \geq 2$ . The image of the 2-primary stable  $J$ -homomorphism in the  $(8j - 1)$ -stem is cyclic of order  $2^{v(8j) + 1}$ . Any element of order  $2^e$  has sphere of origin  $d(e + 2) - 2$  and Adams filtration  $4j + 1 - e$ .*

† Both authors were supported by National Science Foundation Research Grants.

Theorem 1.1 holds also for  $j=1$  except that the sphere of origin for the generator is 8 rather than 9.

The order of the image of  $J$  was established up to a factor of 2 by Adams in [2], and the complete answer was given by the proof of the Adams conjecture in [39] and [34]. The point of our proof here is that it involves only homotopy-theoretic methods, and completes a program devised by the second author in the late 1960s.

“Adams filtration” in the second part of the theorem refers to filtration in the  $(HF_2)$ -Adams spectral sequence (ASS). It is the maximal number of (stable) maps trivial in  $H^*(\ ; F_2)$  which can be composed to yield the given (stable) map. Throughout the paper,  $F_2$  denotes the field with 2 elements.

The sphere of origin of an element  $\alpha \in \pi_i(S^0)$  is the smallest  $n$  such that there is a map  $S^{i+n} \rightarrow S^n$  which suspends to  $\alpha$ . The sphere-of-origin part of 1.1 was stated as [27, 8.4] and [25, 4.4], but the arguments there were incomplete. Adjoining the above definition of  $J$  yields the commutative diagram

$$\begin{array}{ccc} \pi_i(SO(n)) & \xrightarrow{J_n} & \pi_i(\Omega^n S^n) \\ \downarrow i_{SO,n} & & \downarrow i_{S,n} \\ \pi_i(SO) & \xrightarrow{J} & \pi_i(\Omega^\infty S^\infty). \end{array}$$

The sphere of origin of an element in  $\text{im}(J)$  is the smallest  $n$  such that it is in  $\text{im}(i_{S,n})$ . The element in  $\pi_i(\Omega^n S^n)$  will not generally be in  $\text{im}(J_n)$ . The  $SO(n)$ -of-origin for an element of  $\pi_i(SO)$  is  $2j+1$  if  $i=4j-1 > 15$  by [7]. If  $i \equiv 0, 1 \pmod{8}$ , the  $SO(n)$  of origin is 6; this is proved in [18].

A key ingredient in our proof is *bo*-resolutions ([28], [15], [22]). They allow us to prove the following modification of a folk-theorem of the second author, which allows us to prove that certain maps are in the image of  $J$ .

**THEOREM 1.2.** *If  $\alpha \in \pi_i(S^0)$  has Adams filtration greater than  $\frac{3}{10}i + 4 + v(i+2) + v(i+1)$  and satisfies  $i_{J_*}(\alpha) = 0 \in \pi_i(J)$ , then  $\alpha = 0$ .*

Here  $S^0$  is the sphere spectrum, and  $i_J: S^0 \rightarrow J$  is the inclusion into the 2-primary connected  $J$ -spectrum, which will be defined in Section 2. These theorems and the well-known upper edge results of [3] yield the following corollary.

**COROLLARY 1.3.** *Suppose  $\alpha \in \text{Ext}_A^{s,t}(F_2, F_2)$  is an infinite cycle and not a boundary in the ASS, and*

$$s \geq \frac{3}{10}(t-s) + 4 + v(t-s+2) + v(t-s+1).$$

*Then  $\alpha$  detects an element in the image of the stable  $J$ -homomorphism or an Adams  $\mu_j$  element ([2, IV]), and one of the following is true:*

$$\begin{array}{lll} t-s=8j-1 & \text{and} & 4j-v(8j) \leq s \leq 4j \\ t-s=8j & \text{and} & s=4j-1 \\ t-s=8j+1 & \text{and} & 4j \leq s \leq 4j+1 \\ t-s=8j+2 & \text{and} & s=4j+2 \\ t-s=8j+3 & \text{and} & 4j+1 \leq s \leq 4j+3. \end{array}$$

This says that above a line of slope  $3/10$  plus “blips”, in the usual  $(t-s, s)$  coordinates, the  $E_\infty$ -term of the ASS is completely known and very small. The  $E_2$ -term is of course very

large in this range, at least for  $t-s > 50$ . See, e.g., [40] or [32], where it was shown that in our range there are groups  $\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$  of arbitrarily large dimension. This line above which  $E_\infty$  is known can probably be lowered close to slope  $1/5$ , but for technical reasons  $3/10$  is the best we can do now. By methods similar to those of this paper, it can be proved that in the ASS for the mod-2 Moore space all that survives above a  $1/5$ -line is two sequences of “lightning flashes”; the technical problems present for the sphere are not present for the Moore space.

For completeness, we mention that the sphere of origin for elements of the stable image of  $J$  not covered by 1.1 is:

- 3 in the  $8j$ -stem
- 2 in the  $(8j+1)$ -stem
- 2 element of order 2 in the  $(8j+3)$ -stem
- 3 element of order 4 in the  $(8j+3)$ -stem
- 5 element of order 8 in the  $(8j+3)$ -stem.

This was proved in [12] using the upper edge of the unstable ASS. The unstable ASS also allows us to prove that our desuspensions of generators of the stable image of  $J$  also have order  $2^{v(8j)+1}$  in their (unstable) homotopy group.

Unpublished work of Crabb ([9]) dealing with the sphere of origin of the image of the stable  $J$ -homomorphism is discussed briefly in [10]. He utilizes the Adams conjecture, which we have gone to great lengths to avoid.

Theorem 1.2 will be proved in Section 6. In Section 5 we will provide results about the  $bo$ -resolution necessary in this proof, and will make some minor corrections of results of [22] regarding the  $bo$ -ASS.

## 2. OUTLINE OF PROOF OF THEOREM 1.1

We recall the definition of the 2-primary connected  $J$ -spectrum, which plays a central role in our proof. Let  $bo$  (resp.  $bsp$ ) denote the spectrum for connective orthogonal (resp. symplectic)  $K$ -theory localized at 2. The Bott map  $BSp[8k] \rightarrow \Omega^4 BO[8k+4]$  induces an equivalence  $\Sigma^4 bsp \rightarrow bo[4]$ . Here  $X[k]$  denotes the space or spectrum obtained from  $X$  by killing  $\pi_i(\ )$  for  $i < k$ . Thus

$$\pi_i(bo) \approx \begin{cases} \mathbb{Z}_{(2)} & \text{if } i \equiv 0 \pmod{4} \text{ and } i \geq 0 \\ \mathbb{Z}/2 & \text{if } i \equiv 1, 2 \pmod{8} \text{ and } i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_i(\Sigma^4 bsp) \approx \begin{cases} \pi_i(bo) & \text{if } i \geq 4 \\ 0 & \text{if } i < 4 \end{cases}$$

$\mathbb{Z}_{(2)}$  denotes the integers localized at 2, i.e. rational numbers with odd denominators.

One can construct a map  $\theta: bo \rightarrow \Sigma^4 bsp$  nontrivial in  $H^4(\ ; \mathbb{F}_2)$  in a number of ways. The most familiar involves the Adams operation  $\psi^3 - 1$ , but in [31] other constructions were given, and it was shown that all such maps induce the same homomorphism in  $\pi_*(\ )$ , up to units in  $\mathbb{Z}_{(2)}$ . One construction calculates the abelian group of maps  $[bo, bo]$  by the ASS, deducing the existence of infinitely many such maps. Another construction utilizes the splitting

$$bo \wedge bo \simeq bo \vee \Sigma^4 bsp \vee W$$

to obtain the map

$$S^0 \wedge bo \hookrightarrow bo \wedge bo \xrightarrow{\text{pinch}} \Sigma^4 bsp.$$

In [31, 2.4] it was shown that for any such map  $\theta$  the composite  $bo \xrightarrow{\theta} \Sigma^4 bsp \xrightarrow{\pi} bo$  induces in  $\pi_{4i}(\quad)$  multiplication by  $8i$  times a unit in  $\mathbf{Z}_{(2)}$ . The proof there used an (essentially combinatorial) analysis of the abelian group of maps from the  $i$ -fold smash product of  $S^0 \cup_v e^4$  into  $bo$  to deduce the analogous result in homology, and then applied the Hurewicz theorem.

The spectrum  $J$  is defined to be the fiber of  $\theta: bo \rightarrow \Sigma^4 bsp$ , and the homotopy sequence of this fibration shows that for  $i > 0$

$$\pi_{4i-1}(J) \approx \mathbf{Z}/2^{v(8i)}.$$

The following result could be deduced from [2, IV, 7.16]. We will give another proof in Section 3. A third proof, utilizing a simple and direct construction involving the associated infinite loop spaces, appears as [17, 2.10].

PROPOSITION 2.1. *The composite*

$$\pi_{8j-1}(SO) \xrightarrow{J} \pi_{8j-1}(S^0) \xrightarrow{i_*} \pi_{8j-1}(J)$$

*is surjective.*

Thus the order of the image of  $J$  is at least as large as claimed in 1.1. Much of our work is directed toward proving indirectly that  $i_*$  is injective on  $\text{im}(J)$ .

Another ingredient in the proof of 1.1 is the construction of maps with the desired filtration and sphere of origin and which project to appropriate elements of  $\pi_{8j-1}(J)$ . This was done in [27, 7.15 and p. 107], and another discussion of these ideas was presented in [16, pp. 138–9]. The result needed here is stated below; its proof will also be discussed in Section 3. Let  $P_n$  and  $P_n^m$  denote the stunted real projective spaces  $RP^\infty/RP^{n-1}$  and  $RP^m/RP^{n-1}$ , respectively. In 3.1 we define a family of spaces  $P_n^i$  formed as slight modifications of stunted real projective spaces, sometimes after one suspension. These are just skeleta of the spaces called  $Q_n$  in [16, 3.4] and  $C(n)$  in [27, 7.15].

PROPOSITION 2.2. (i) *Suppose  $v(8j) \geq 4e + \varepsilon$ ,  $0 \leq \varepsilon \leq 3$ . There exist stable maps (which are, in fact, honest maps of the double suspensions of these spaces; see 3.1)*

$$\begin{aligned} \varepsilon=3 \quad S^{8j-1} &\xrightarrow{d} P_{8j-8e-7}^{8j} \xrightarrow{l} P_1^{8e+8} \\ \varepsilon=2 \quad S^{8j-1} &\xrightarrow{d} P_{*8j-8e-6}^{8j} \xrightarrow{f} P_{8j-8e-7}^{8j-2} \xrightarrow{l} P_1^{8e+6} \\ \varepsilon=1 \quad S^{8j-1} &\xrightarrow{d} P_{*8j-8e-5}^{8j} \xrightarrow{l} P_{*3}^{8e+8} \xrightarrow{f} P_{*2}^{8e+6} \xrightarrow{f} P_1^{8e+5} \\ \varepsilon=0 \quad S^{8j-1} &\xrightarrow{d} P_{8j-8e-1}^{8j} \xrightarrow{f} P_{*8j-8e-5}^{8j-2} \xrightarrow{f} P_{*8j-8e-6}^{8j-3} \xrightarrow{f} P_{8j-8e-7}^{8j-4} \xrightarrow{l} P_1^{8e+4} \end{aligned}$$

*such that  $d$  is nontrivial in  $H^*(\quad; \mathbf{F}_2)$ ,  $f$  has filtration 1, and  $l$  has filtration  $4j - 4e - 4$ .*

(ii) *If  $c \in \pi_{8j-1}(P^{n-1})$  denotes any of the composites of (i) and  $i: P^{n-1} \rightarrow P^\infty$  the inclusion, then  $i_*(c)$  is an element of order  $2^{4e+\varepsilon+1}$  in  $J_{8j-1}(P^\infty) \approx \mathbf{Z}/2^{v(8j)+1}$ .*

This is related to  $\pi_{8j-1}(J)$  by the following result, also proved in Section 3.

PROPOSITION 2.3. Let  $\lambda: P^\infty \rightarrow S^0$  denote the map of spectra induced by the maps

$$(2.4) \quad \Sigma^n P^{n-1} \rightarrow \Sigma^n SO(n) \xrightarrow{\mu} S^n.$$

Then  $\lambda_*: J_{8j-1}(P^\infty) \rightarrow \pi_{8j-1}(J)$  is an isomorphism.

Next we need a crude lower bound on the filtration of the image of the  $J$ -homomorphism. This represents a minor modification of the main technical result of [24], which considered the complex  $J$ -homomorphism.

PROPOSITION 2.5. If  $8j < 2^l$ , then the generator of the image of the stable  $J$ -homomorphism in the  $(8j-1)$ -stem has Adams filtration  $\geq 4j+1-l$ .

*Proof.* Let  $d$  be as defined prior to 1.1. The key here is that the natural map  $SO[d(k)] \rightarrow SO[d(k-1)]$  is 0 in  $H^i(-; \mathbb{F}_2)$  for  $i < 2^k - 1$ . This was proved in [23], following earlier related work in [38]. Let  $n = 2^l - 2 \geq 8j + 6$ . Then the map of skeleta

$$p: S^{8j-1} = SO[8j-1]^{(8j)} = SO[d(4j-1)]^{(8j)} \rightarrow SO[d(l-1)]^{(8j)} = SO_n[d(l-1)]^{(8j)}$$

has filtration  $\geq 4j-l$ . Then the composite

$$S^{8j-1+n} \xrightarrow{\Sigma^n p} \Sigma^n SO_n[d(l-1)]^{(8j)} \xrightarrow{\Sigma^n q} \Sigma^n SO_n \xrightarrow{\mu} S^n$$

represents a generator of the stable  $J$ -homomorphism in the  $(8j-1)$ -stem and has filtration  $\geq 4j-l+1$ , since  $\mu$  has filtration  $\geq 1$ .  $\blacksquare$

Now we can prove Theorem 1.1. Let  $c: S^{8j-1+n} \rightarrow \Sigma^n P^{n-1}$  be an iterated suspension of one of the composites of 2.2(i), and  $\lambda_{n-1}$  the map of (2.4). Using 2.2(i) and 2.3, and that  $\lambda$  has filtration 1, the stable class  $h = [\lambda_{n-1} \circ c] \in \pi_{8j-1}(S^0)$  has the sphere of origin and filtration desired for an element in the image of  $J$  whose order equals the order of  $i_{J*}(h)$  in  $\pi_{8j-1}(J)$ . Indeed, in the notation of 2.2,  $i_{J*}(h)$  has order  $2^{4e+\varepsilon+1}$ , filtration  $4j-4e-\varepsilon$ , and sphere of origin

$$8e + \begin{cases} 9 & \text{if } \varepsilon = 3 \\ 7 & \text{if } \varepsilon = 2 \\ 6 & \text{if } \varepsilon = 1 \\ 5 & \text{if } \varepsilon = 0. \end{cases}$$

Here it was necessary that the maps of 2.2(i) be defined after a number of suspensions no larger than the dimension of the target projective space plus 1. It is clear that at least one suspension must have been required, for otherwise the stable image of  $J$  would be in the image of the following composite

$$\pi_{8j-1}(P^{n-1}) \rightarrow \pi_{8j-1}(SO(n)) \rightarrow \pi_{8j-1}(SO) \xrightarrow{J} \pi_{8j-1}^s,$$

where the first and last groups are finite, while the penultimate group is infinite cyclic.

Let  $g \in \pi_{8j-1}(S^0)$  denote the stable class of the map defined in 2.5, a generator of the stable image of  $J$ . By 2.1  $i_{J*}(g)$  is a generator of the cyclic 2-group  $\pi_{8j-1}(J)$ , and by 2.2(ii) and 2.3  $i_{J*}(h)$  is a certain multiple,  $k$ , of this generator;  $k$  satisfies  $v(k) = v(8j) - 4e - \varepsilon$ . Thus  $i_{J*}(kg - h) = 0$ . If  $l$  denotes the smallest integer such that  $2^l > 8j$ , then the Adams filtration of  $kg - h$  satisfies

$$\begin{aligned} \text{AF}(kg - h) &\geq \min(\text{AF}(g) + v(k), \text{AF}(h)) \\ &\geq \min(4j+1-l + v(8j) - 4e - \varepsilon, 4j - 4e - \varepsilon) \\ &\geq 4j+1-l, \end{aligned}$$

and one easily checks that if  $j > 6$ , this filtration satisfies the hypothesis of 1.2. Thus by 1.2,  $kg - h = 0$ , so that  $kg$  has the filtration and sphere of origin desired of an element of its order in the stable image of  $J$ . Since  $\text{Ext}_{\mathcal{A}}^{s, s+8j-1}(\mathbf{F}_2, \mathbf{F}_2) = 0$  for  $s > 4j$  ([3]), the ASS implies that  $2^{v(8j)+1}g = 0$ , and with 2.1 establishes the order in 1.1. If  $j \leq 6$ , then  $kg - h$  is within 5 of the upper edge in the ASS, and as in [29, 2.7] the ASS in this region is well understood, allowing us to deduce  $kg - h = 0$ , from which 1.1 follows as before.

One detail remains—showing that the sphere of origin of the generator of the image of  $J$  can be no smaller than that claimed in 1.1. This will be proved in Section 4 using the James–Hopf maps.

### 3. PROOFS OF 2.1, 2.2, AND 2.3

The proof of 2.1 is by induction on  $j$ . As in [19], one easily shows that for  $t$  odd with  $t > 4j + 5$  the map  $\cdot 16$  on  $P_t^{8j+8}$  compresses into  $P_t^{8j}$ , sending  $P_t^{8j+6}$  into  $P_t^{8j-1}$ , yielding a commutative diagram

$$\begin{array}{ccc} P_t^{8j+8} & \xrightarrow{l_1} & P_t^{8j} \\ \downarrow c_1 & & \downarrow c_2 \\ P_{8j+7}^{8j+8} & \xrightarrow{l_2} & S^{8j}, \end{array}$$

where  $c_1$  and  $c_2$  are collapse maps. One easily verifies that for induced homomorphisms in  $\widetilde{KO}(\quad)$ ,  $c_2^*$  sends the generator of an infinite cyclic group to the class of order 2 in a cyclic 2-group,  $l_1^*$  is a monomorphism of cyclic 2-groups, and  $c_1^*$  sends  $\mathbf{Z}/2$  injectively to the class of order 2. Hence  $l_2^*$  is surjective.

A similar deduction can be made for induced homomorphisms in the generalized cohomology group  $J^1(\quad)$ , the only difference being that  $J^1(S^{8j}) \approx J^0(S^{8j-1}) \approx \pi_{8j-1}(J)$  is a finite cyclic 2-group. Since these calculations are less familiar, we discuss them at the end of this section.

There is a natural transformation  $T_X: \widetilde{KO}(\Sigma X) \rightarrow J^0(X)$  defined by applying  $[X, \quad]$  to the composite

$$SO \xrightarrow{J} \Omega^\infty S^\infty \xrightarrow{i_j} \Omega^\infty J,$$

where the first map is adjoint to the maps  $\mu$  of Section 1. Its homotopy homomorphism is the stable  $J$ -homomorphism. Here  $\Omega^\infty$  takes the underlying loop space of the spectra  $S^0$  and  $J$ .

We apply  $T$  to the desuspension of the map  $l_2$  above, assuming by induction that  $T_{S^{8j-1}}$  is surjective, and deduce that the middle homomorphism below is the nonzero homomorphism of groups of order 2.

$$\begin{array}{ccc} \widetilde{KO}(S^{8j}) & \xrightarrow{T_{S^{8j-1}}} & J^0(S^{8j-1}) \\ \downarrow l_2^* & & \downarrow l_2^* \\ \widetilde{KO}(P_{8j+7}^{8j+8}) & \xrightarrow{T} & J^0(\Sigma^{-1} P_{8j+7}^{8j+8}) \\ \downarrow c^* & & \downarrow c^* \\ \widetilde{KO}(S^{8j+8}) & \xrightarrow{T_{S^{8j+7}}} & J^0(S^{8j+7}) \end{array}$$

Since both homomorphisms  $c^*$  are surjective, we deduce the lower  $T$  is surjective, extending the induction, and proving 2.1.

In order to construct the maps of 2.2(i), we need to modify stunted real projective spaces at their top and/or bottom. To do this, we modify the notation of [27, p. 104] and [16, 3.4] in the following result. We use the standard notation that for a spectrum  $X$ ,  $X^{<m>}$  is the spectrum formed by killing elements of  $\text{Ext}_{A_1}^s(H^*X, \mathbb{F}_2)$  with  $s < m$ .

**PROPOSITION 3.1.** *For positive integers  $b, t, M, N$  with  $N \equiv 2, 3 \pmod{4}$  and  $M \equiv 0, 1 \pmod{4}$ , there exist spectra  $P_{*N}^t$ ,  $P_b^{*M}$ , and  $P_{*N}^{*M}$  satisfying*

(i) *There is a commutative diagram in which vertical maps are cofibrations, and nonvertical maps have filtration 1.  $T$  may be  $t$  or  $*M$ .*

$$(3.2) \quad \begin{array}{ccccc} & \Sigma^{4n}(S^5 \cup_2 e^6) & & \Sigma^{4n}(S^2 \cup_\eta e^4 \cup_2 e^5) & \\ & \downarrow p & & \downarrow q & \\ P_{4n+5}^T & \xrightarrow{f_1} & P_{4n+3}^T & \xrightarrow{f_3} & P_{4n+2}^T \\ \downarrow i_2 & & \downarrow i_1 & & \downarrow i_3 \\ P_{4n+7}^T & \xrightarrow{j_1} & P_{*(4n+3)}^T & \xrightarrow{j_3} & P_{*(4n+2)}^T \\ & & & & \nearrow f_2 \\ & & & & P_{4n+1}^T \xrightarrow{j_2} P_{4n+1}^T \xrightarrow{j_4} P_{4n-1}^T \end{array}$$

Dually, there is a diagram in which  $S$  may be  $b$  or  $*N$ .

(3.3)

$$(3.3) \quad \begin{array}{ccccccc} & \Sigma^{4m}(S^1 \cup_2 e^2) & & \Sigma^{4m}(S^2 \cup_2 e^3 \cup_\eta e^5) & & & \\ & \uparrow & & \uparrow & & & \\ P_S^{4m+2} & \xleftarrow{f'_1} & P_S^{4m+4} & \xleftarrow{f'_3} & P_S^{4m+5} & & \\ \uparrow & & \uparrow & & \uparrow & & \\ P_S^{4m} & \xleftarrow{j'_1} & P_S^{*(4m+4)} & \xleftarrow{j'_3} & P_S^{*(4m+5)} & \xleftarrow{f'_4} & P_S^{4m+6} \xleftarrow{f'_4} P_S^{4m+8} \end{array}$$

(ii) *If  $P_{S_1}^T \rightarrow P_{S_2}^T$  is any of the nonvertical maps in (3.2) and  $P_{S_1}^{T_1} \rightarrow P_{S_2}^{T_2}$  any of the nonvertical maps in (3.3), then there is a filtration 1 map  $P_{S_1}^{T_1} \xrightarrow{l} P_{S_2}^{T_2}$  such that the induced morphism*

$$\text{Ext}_{A_1}^{s,t}(H^* P_{S_1}^{T_1}, \mathbb{F}_2) \rightarrow \text{Ext}_{A_1}^{s+1,t+1}(H^* P_{S_2}^{T_2}, \mathbb{F}_2)$$

*is nontrivial wherever both groups are nonzero. If both of the maps were selected from the bottom row, then  $l \wedge bo$  lifts to an equivalence*

$$P_{S_1}^{T_1} \wedge bo \rightarrow (P_{S_2}^{T_2} \wedge bo)^{<1>}.$$

(iii)  $P_S^T$  exists as a space unless  $T$  is starred or  $S = *2$ . In those cases, it exists after one suspension, i.e., there is a space  $\sigma P_S^T$  whose suspension spectrum is  $\Sigma P_S^T$ . The maps of (i) and (ii) exist as maps of spaces after 2 suspensions.

**Remark.** For  $N \equiv 2, 3 \pmod{4}$ ,  $P_{*N}^m$  is the  $m$ -skeleton of  $Q_N$  of [16, 3.4]. It should be thought of as a modified version of the stunted real projective space  $P_N^m$ .

*Proof.* We first construct (3.2) when  $T$  is unstarred, by the method of [27, 7.15]. Unless  $4n + \varepsilon = 2$ , the maps  $p$  and  $q$  exist as actual maps of spaces since their construction depends on calculations (from [30]) of  $\pi_*(P_{4n+\varepsilon})$  in the stable range, and hence  $P_{*(4n+\varepsilon)}^T$  exists as an actual space. If  $4n + \varepsilon = 2$ , we must suspend once in order that the domain space of  $q$  exist.

The maps  $f_i$  are extensions of  $\cdot 2$ , modified by an attaching map in the case of  $f_3$ . Since  $\cdot 2$  requires a suspension for its definition, so do the maps  $f_i$ . In the case  $4n + \varepsilon = 2$ , we have already suspended. The maps  $j_i$  are extensions over mapping cones of maps which are trivial by calculations of homotopy groups in the stable range, and so exist after the single suspension that was required for  $f_i$ .

The top vertical maps in (3.3) exist as actual maps of spaces (even if  $S$  is starred), again by calculations in the stable range. Their mapping cone is the space  $\sigma P_S^{*(4m+\varepsilon)}$  of (iii), a space whose suspension spectrum is  $\Sigma P_S^{*(4m+\varepsilon)}$ . The maps  $f'_i$  in (3.3) are compressions of  $\cdot 2$ , possibly modified, and so one suspension is required for their existence. The maps along the bottom of (3.3) exist by the following standard lemma. Condition (i) will usually apply, but not if  $S = 1$ , and so a suspension may be required.

LEMMA 3.4. *If  $A \xrightarrow{f} B \xrightarrow{g} C$  is a cofibration, and  $h: W \rightarrow B$  is such that  $g \circ h$  is nulhomotopic, then there is a map  $j: W \rightarrow A$  such that  $f \circ j \simeq h$  provided either*

- (i)  *$A$  is  $(a-1)$ -connected,  $C$  is  $(c-1)$ -connected, and  $\dim(W) \leq a + c - 2$ ;*
- or (ii) *All spaces and maps are suspensions.*

Next one does the analysis of (3.2) again if  $T$  is starred. Here 2 suspensions are required, one for  $P_S^T$  to exist, and another for it to have a  $\cdot 2$ .

The construction in (ii) is similar. If neither  $S_1$  nor  $S_2$  is starred, consider the diagram

$$\begin{array}{ccc} P_{S_2}^{S_1-1} & & \\ \downarrow i & & \\ P_{S_2}^{T_1} & \xrightarrow{l} & P_{S_2}^{T_2} \\ \downarrow c & & \\ P_{S_1}^{T_1} & & \end{array}$$

where  $l$  is a filtration-1 map constructed in (3.3). One verifies, using the charts of [30], that  $l \circ i$  is null, so that the desired extension over  $P_{S_1}^{T_1}$  exists.

Finally, the construction in (ii) when  $S_1$  and/or  $S_2$  is starred follows from the maps just constructed in the same way that the maps  $j_i$  followed from  $f_i$  in (3.2). For example, the square below can be filled in because  $i_3 \circ l \circ p \simeq *$  by charts of [30].

$$\begin{array}{ccc} S^{4n+5} \cup_2 e^{4n+6} & & \\ \downarrow p & & \\ P_{4n+3}^{T_1} & \xrightarrow{l} & P_{4n+2}^{T_2} \\ \downarrow & & \downarrow i_3 \\ P_{*(4n+3)}^{T_1} & & P_{*(4n+2)}^{T_2} \end{array}$$

One can calculate  $\pi_*(P_S^T \wedge bo)$  as in [13, §3] using the  $A_1$ -structure of  $H^*(P_S^T)$ . Here and elsewhere  $A_1$  is the subalgebra of the mod 2 Steenrod algebra generated by  $Sq^1$  and  $Sq^2$ . In [16, 3.5] the reader will find a start toward the explicit correspondence between the  $A_1$ -modules  $H^*(P_S^T)$  encountered here and the  $A_1$ -modules  $\mathcal{P}_{ijn}$  of [13, 3.6] whose  $\text{Ext}_{A_1}$  is given in [13, 3.10]. There are clearly maps  $P_{S_1}^{T_1} \wedge bo \rightarrow (P_{S_2}^{T_2} \wedge bo)^{\langle 1 \rangle}$ , and their construction shows that the homomorphism in  $\pi_*(\ )$  is an isomorphism on the first few groups. That it is iso on all the groups then follows from the  $\pi_*(bo)$ -module structure.



The proof of 3.1 is now complete, and 2.2(i) is immediate. The maps  $f$  there are appropriate maps  $l$  from 3.1(ii), and maps  $l$  of 2.2(i) are composites of  $4j-4e-4$  maps of 3.1(ii). Note that a composite of 4 of the maps along the bottom of (3.2) or (3.3) ends at a space of the same type with which it started, so that long composites can be formed. For regular stunted projective spaces, these are just the filtration 4 maps of [16, 2.1]. The degree 1 map  $d$  of 2.2 is [16, 3.6], an easy consequence of the splitting of the bottom cell of the Thom complex of a trivial vector bundle.

For use in the above proof of 2.1 and the proofs of 2.2(ii) and 2.3, we review the calculation of  $J_*(X)$ . If  $E$  is a spectrum such as  $J$ ,  $bo$ , or  $bsp$ , we denote by  $E_*(X)$  the generalized homology groups  $\pi_*(X \wedge E)$ .  $J_*(X)$  is calculated from a chart with

$$E_1^{s,t}(X) = \text{Ext}_{A_1}^{s,t}(H^* X, \mathbf{F}_2) \oplus \text{Ext}_{A_1}^{s-1,t}(H^* X \otimes B, \mathbf{F}_2),$$

where  $B = \Sigma^4 \langle 1, Sq^2, Sq^3 \rangle$ . These charts are of the type illustrated in [27] and [16], having as  $(x, y)$  coordinates  $(t-s, s)$ . They are not Ext charts, but are the charts corresponding to a resolution. If  $X$  is a stunted projective space, there are nonzero (as long as elements are present) differentials  $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$  for  $r \geq 1$  on towers in degree  $t-s$  satisfying  $v(t-s+1) = r+1$ . From this, one easily obtains the result for  $J_{8j-1}(P^\infty)$  in 2.2ii, and the following result, where  $d$  is as defined in Section 1.

**PROPOSITION 3.5.** *An element of order  $2^e$  in  $J_{8j-1}(P^\infty)$  is in the image from  $J_{8j-1}(P^{n-1})$  if and only if  $n \geq d(e+2)-2$  or  $j=1$ ,  $e=4$ , and  $n=8$ .*

The order in  $J_{8j-1}(P^\infty)$  of the elements described in 2.2ii is immediate from these charts and the injectivity in  $bo_*( )$  of the maps constructed in 3.1.

*Proof of 2.3.* Let  $R$  denote the mapping cone of  $\lambda: P^\infty \rightarrow S^0$ . In [31] it was shown that  $R \wedge bo \simeq \bigvee_{i \geq 0} \Sigma^{4i} H\mathbf{Z}$  is a wedge of Eilenberg–MacLane spectra. In the exact sequence

$$0 \rightarrow J_{8j-1}(P^\infty) \xrightarrow{\lambda_*} J_{8j-1}(S^0) \rightarrow J_{8j-1}(R) \xrightarrow{\delta} J_{8j-2}(P^\infty) \rightarrow 0$$

$\delta$  is an isomorphism of  $\mathbf{Z}/2^{v(j)+1}\mathbf{s}$ , and hence  $\lambda_*$  is an isomorphism. ■

We have now completed the proof of 2.2, and all that remains for 2.1 is to verify claims made about homomorphisms in  $J^1( )$  in the second paragraph of this section. After dualizing to yield a statement about the more familiar  $J_*( )$ , we find that we must prove that in the diagram

$$\begin{array}{ccc} J_{-2}(P_{-8j-9}^{-t-1}) & \xleftarrow{l_*} & J_{-2}(P_{-8j-1}^{-t-1}) \\ i_{1*} \uparrow & & \uparrow i_{2*} \\ J_{-2}(P_{-8j-9}^{-8j-8}) & & J_{-2}(S^{-8j-1}), \end{array}$$

where  $l$  is a filtration 4 map of the type constructed in 3.1,  $\text{im}(i_{1*})$  and  $\text{im}(i_{2*})$  are both generated by the element of order 2, and  $l_*$  is injective. Here negatively graded stunted projective spaces are as defined in [4] or [16]. They provide a convenient context for duality, although they can be avoided by adding a large 2-power to all indices.

The above method for calculating  $J_*( )$  works for negatively graded projective spaces, too. The differential from  $t-s=-2$  to  $t-s=-1$  is 0, since 0 is infinitely 2-divisible. (This can also be seen by the James-periodicity definition of negatively indexed projective spaces.)

Thus  $bsp_{-5}(P_{-k}^{-n}) \rightarrow J_{-2}(P_{-k}^{-n})$  is an isomorphism, at least if  $k$  is odd and  $n$  even, and so it suffices to prove the analogous facts about  $bsp_{-5}( )$ . The injectivity of  $l_*$  in  $bo_*( )$  for the

filtration 4 maps noted in 3.1ii easily implies the analogue for  $bsp_*$  ( ). Indeed, using the notation of 3.1ii,

$$(P_{S_1}^{T_1} \wedge bsp)^{\langle 1 \rangle} \rightarrow (P_{S_2}^{T_2} \wedge bsp)^{\langle 2 \rangle}$$

is an equivalence.

Finally, an easy calculation (see charts below, which can be extended to the right in an obvious way;  $\text{im}(j_*)$  and  $\text{im}(i_*)$  are indicated in boxes) shows that in

$$bsp_{-5}(S^{-8j-9}) \xrightarrow{j_*} bsp_{-5}(P_{-8j-9}^{-8j-8}) \xrightarrow{i_*} bsp_{-5}(P_{-8j-9}^{-t-1})$$

$j_*$  is surjective, and  $\text{im}(i_*)$  is generated by the element of order 2.

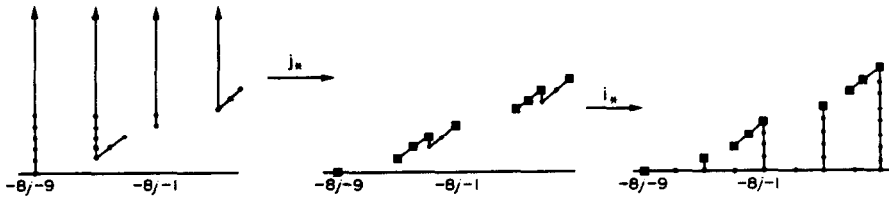


Fig. 1.

#### 4. PROOF THAT THE SPHERE OF ORIGIN CAN BE NO LOWER THAN CLAIMED IN 1.1

Our proof uses the following result, to which many have contributed. Its proof will be discussed after we show how it is applied. Let  $QX = \Omega^\infty \Sigma^\infty X$ . Then  $QS^0 = \Omega^\infty S^\infty$ , and  $Q_0 S^0 = \Omega_0^\infty S^\infty$  is the component of the constant map.

**THEOREM 4.1.** *There is a commutative diagram*

$$\begin{array}{ccc} \Omega_0^n S^n & \xrightarrow{j_*} & QP^{n-1} \\ \downarrow & & \downarrow \\ \Omega_0^\infty S^\infty & \xrightarrow{j} & QP^\infty \xrightarrow{\Omega^\infty s} Q_0 S^0 \end{array}$$

such that the bottom composite is a 2-local homotopy equivalence, and  $\Omega^\infty s = \Omega^\infty \lambda \circ \Omega^\infty h$  with  $\lambda$  as in 2.3 and  $h$  a self-equivalence of  $\Sigma^\infty P^\infty$ .

*Remark.*  $QP^{n-1} = \Omega^\infty \Sigma^\infty P^{n-1}$  should not be confused with common notation for quaternionic projective space.

We extend 4.1 to a commutative diagram

$$(4.2) \quad \begin{array}{ccccccc} \Omega_0^n S^n & \xrightarrow{j_*} & QP^{n-1} & \longrightarrow & \Omega^\infty(P^{n-1} \wedge J) & \longrightarrow & \Omega^\infty(P^\infty \wedge J) \\ \downarrow & & \downarrow & & & & \downarrow s \wedge J \\ \Omega_0^\infty S^\infty & \xrightarrow{j} & QP^\infty & \xrightarrow{\Omega^\infty s} & Q_0 S^0 & \xrightarrow{i_J} & \Omega^\infty J \end{array}$$

In the composite which defined  $T_X$  in our proof of 2.1,  $i_J$  could have been replaced by the composite along the bottom of (4.2). Hence, our proof of 2.1 shows that the bottom composite sends the image of  $J$  onto  $\pi_{8j-1}(J)$ , and by 1.1 it does so isomorphically.

Our proof of 2.3 applied to  $s$  just as well as  $\lambda$ , since the splitting of  $R \wedge bo$  in [31] only required the  $A$ -module structure. Thus the right vertical map in (4.2) is a weak equivalence, and so (4.2) shows that if an element of order  $2^e$  in the image of the stable  $J$ -homomorphism has sphere of origin  $\leq n$ , then  $J_{8j-1}(P^{n-1}) \rightarrow J_{8j-1}(P^\infty)$  maps onto elements of order  $2^e$ , and hence by 3.5  $n \geq d(e+2)-2$ , establishing sharpness of the sphere of origin in 1.1. The exceptional behavior for the generator if  $j=1$  noted after 1.1 follows also from 3.5.

We now make some comments on the proof of 4.1. The equivalence in the bottom row is the Kahn–Priddy theorem ([20]), the existence of maps  $j_n$  was first proved by Snaith ([37]), and Segal ([35]) proved a result like 4.1 except that his map  $QP^\infty \rightarrow Q_0S^0$  was not known to be an infinite loop map. In [8], more geometric descriptions of maps  $j_n, j$ , (called James–Hopf maps because of their similarity to James’ 1957 construction of the Hopf invariant), and an infinite loop map  $s: QP^\infty \rightarrow Q_0S^0$  were given. By [5], the infinite loop map  $s$  could be preceded by a self-equivalence of  $QP^\infty$  to yield  $Q\lambda$ . Finally, Kuhn ([21]) showed compatibility of the maps  $j_n$  as  $n$  increases, so that  $j$  could be taken as the union of these maps. Underlying the constructions of [37] and [8] is May’s model ([33]) of  $\Omega^n \Sigma^n X$  as little cubes in  $I^n$  labeled with points of  $X$ . This allowed various combinatorial manipulations used in defining the maps  $j_n$  and  $s$ .

In [6], the  $e$ -invariant was used to obtain lower bounds for the sphere of origin of the image of  $J$  which were almost as strong as those of 1.1.

This sphere-of-origin result is closely related to the Barratt–Mahowald conjecture (see [36]) that for  $m \geq 3$  the 2-primary exponent of  $\pi_*(S^{2m+1})$  is

$$\begin{cases} 2^m & \text{if } m \equiv 0, 3 \pmod{4} \\ 2^{m-1} & \text{if } m \equiv 1, 2 \pmod{4}. \end{cases}$$

This exponent is defined to be the minimum of the orders of all elements of  $\pi_i(S^{2m+1})$  for  $i > 2m+1$ . Our desuspensions of the image-of- $J$  generators, i.e. the composites

$$S^{8j-1+2m+1} \xrightarrow{c} \Sigma^{2m+1} P^{2m} \xrightarrow{\lambda} S^{2m+1}$$

of 2.2 and 2.3 give elements of the desired order. A proof of the Barratt–Mahowald conjecture would give another proof that the odd sphere of origin can be no smaller than claimed in 1.1.

Yet another approach to the sphere of origin is via the EHP sequence. The unstable ASS (see [11, 11.9]) shows that if  $\alpha$  as constructed in Section 2 is a maximal filtration desuspension of the generator of the image of  $J$  in the  $(8j-1)$ -stem to the sphere  $S^{n_0}$  of that construction, then the Hopf invariant  $H(\alpha)$  is the element of maximal Adams filtration in the  $(8j-n_0)$ -stem, and so this map desuspends no farther. This does not directly preclude the possibility that the generator of the stable image of  $J$  might desuspend to a low-filtration element on this sphere which desuspends even lower. However, the argument at the beginning of this section does preclude that.

## 5. $bo$ -RESOLUTIONS

In this section we apply the results of [22] to obtain a useful but technical result about the relationship of Adams filtration with liftings in the  $bo$ -resolution. We also make a few minor corrections to results of [22], which affect their calculations of the  $bo$ -ASS for  $S^{0\langle i \rangle}$  if  $i \geq 2$ , but not in the basic case  $i=0$ .

For the reader who prefers to skip the delicate details of this section, we begin by stating the main result of this section, which will be applied in the next section to prove 1.2. Let

$S^0 \rightarrow bo \rightarrow \overline{bo}$  denote the standard cofibration, and  $S_s = (\Sigma^{-1} \overline{bo})^s$ , the smash product of  $s$  factors. The  $bo$ -resolution of  $S^0$  is the following tower of cofibrations.

$$\begin{array}{ccccccc} S^0 = S_0 & \xleftarrow{p_0} & S_1 & \xleftarrow{p_1} & S_2 & \xleftarrow{\quad} & \cdots \\ \downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 & & \\ S_0 \wedge bo & & S_1 \wedge bo & & S_2 \wedge bo & & \end{array}$$

In the rest of the paper,  $AF(f)$  refers to the  $(HF_2)$ -Adams filtration of the map  $f$ .

**THEOREM 5.1.** *Let  $f_s: S^n \rightarrow S_s$  be any map from the  $n$ -sphere to the  $s$ -th stage of the  $bo$ -resolution.*

- (i) *If  $s \geq 3$  and  $AF(f_s) = a \geq 2$ , then there is a map  $f_{s+1}: S^n \rightarrow S_{s+1}$  such that  $p_{s-1} \circ p_s \circ f_{s+1} = p_{s-1} \circ f_s$  and*

$$AF(f_{s+1}) \geq \begin{cases} a-2 & \text{if } n+s \equiv 0, 1, 2, 4 \pmod{8} \\ a-1 & \text{if } n+s \equiv 3, 5, 6, 7 \pmod{8}. \end{cases}$$

- (ii) *If  $s=2$  and  $AF(f_s) = a > v(n+2)$ , then there is a map  $f_{s+1}$  as above with  $AF(f_{s+1}) \geq a - v(n+2) - 1$ .*

- (iii) *If  $s=0$  or  $1$  and  $i_{J*}(f_s) = 0 \in \pi_n(J)$ , then there is  $f_{s+1}$  as above with*

$$AF(f_{s+1}) \geq \begin{cases} AF(f_s) - 1 & s=0 \\ AF(f_s) - \max(1, v(n+1) - 1) & s=1. \end{cases}$$

*Remark.* The statements in (ii) involving the 2-power  $v(n+2)$  are very crude; refinements can easily be obtained from the work later in this section.

In order to prove 5.1, we calculate the  $E_2$ -term of the  $bo$ -regular part of the  $bo$ -ASS for  $S^{0\langle i \rangle}$ . These groups were calculated (with minor mistakes if  $s=2$  and  $i \geq 2$ ) in [22, 6.1, 6.6, 6.11], where they were called  $H^{s,i}(\mathcal{F}_i \mathcal{C})$ . We recall some of their notation and basic results.  $\mathcal{F}_i \mathcal{C}^{**}$  is the quotient complex obtained from the chain complex

$$\pi_i(bo \wedge \overline{bo}^{s-1} \wedge S^{0\langle i \rangle}) \rightarrow \pi_i(bo \wedge \overline{bo}^s \wedge S^{0\langle i \rangle}) \rightarrow \pi_i(bo \wedge \overline{bo}^{s+1} \wedge S^{0\langle i \rangle})$$

by dividing out by the homotopy groups of the mod 2 Eilenberg–MacLane summands. Also,  $\mathcal{F}_0 \mathcal{C} = \mathcal{C}$ . The following result is standard, and contained in [22] or [15].

**PROPOSITION 5.2.** *The groups  $\mathcal{F}_i \mathcal{C}^{s,t}$  are 0 if  $t \equiv 3, 5, 6, 7 \pmod{8}$ . If  $t \equiv 1$  or  $2 \pmod{8}$ , then  $\mathcal{F}_i \mathcal{C}^{s,t}$  is a sum of  $\mathbb{Z}_2$ 's, and for every cycle  $\alpha$  in the complex*

$$\mathcal{F}_i \mathcal{C}^{s-1,t} \xrightarrow{d} \mathcal{F}_i \mathcal{C}^{s,t} \xrightarrow{d} \mathcal{F}_i \mathcal{C}^{s+1,t}$$

*with  $AF(\alpha) = a > 0$ , there exists  $\beta$  such that  $\alpha = d(\beta)$  with  $AF(\beta) \geq a - 1$ .*

The case  $t \equiv 0 \pmod{4}$  is by far the most difficult. Here  $\mathcal{F}_i \mathcal{C}^{s,4*}$  is a free  $\mathbb{Z}_{(2)}$ -module in each grading with basis

$$(5.3) \quad \{2^a w^m e_i[\tau_{n_1} | \dots | \tau_{n_s}] \in \mathcal{F}_i \mathcal{C}^{s,4(m+|n|)} : \\ m \geq 0, n_i > 0, a = \max(0, 2'(m - |n|) + \alpha(n) - i)\}$$

where

$$2'N = \begin{cases} 2N & \text{if } N \text{ is even} \\ 2N+1 & \text{if } N \text{ is odd.} \end{cases}$$

Here  $\mathbf{n}=(n_1, \dots, n_s)$ ,  $|\mathbf{n}|=\sum n_i$ , and  $\alpha(\mathbf{n})=\sum \alpha(n_i)$ , with  $\alpha(n_i)$  the number of 1s in the binary expansion of  $n_i$ .  $w$ ,  $\varepsilon_i$ , and  $\tau_n$  have homological/homotopical interpretations; see [22, p. 610]. The boundary homomorphism satisfies

$$d(w^m \varepsilon_i[\tau_{n_1} | \dots | \tau_{n_s}]) = \sum_j \binom{m}{j} w^{m-j} \varepsilon_i[\tau_j | \tau_{n_1} | \dots | \tau_{n_s}] \\ + \sum_{k,j} (-1)^k \binom{n_k}{j} w^m \varepsilon_i[\tau_{n_1} | \dots | \tau_{n_k-j} | \tau_j | \dots | \tau_{n_s}],$$

where  $\binom{m}{j}$  as in [22, p. 598] has properties similar to binomial coefficients.  $D_0$  (resp.  $D_1$ ) refers to the sum of the terms in the formula for  $d$  in which the coefficients  $\binom{m}{j}$  are even (resp. odd).

It was claimed in [22, 6.6] that the groups  $H^{s,4t}(\mathcal{F}_i\mathcal{C})$  are  $\mathbb{F}_2$ -vector spaces for  $s \geq 2$ . We show by example that this need not be true for  $s=2$ , and then will pinpoint the mistake in their proof when  $s=2$ . Their proof is valid if  $s \geq 3$ . Then we will prove as Proposition 5.5 a corrected version of [22, 6.6].

We show  $H^{2,16}(\mathcal{F}_2\mathcal{C}) \approx \mathbb{Z}/4$ . A matrix of the homomorphism  $\mathcal{F}_2\mathcal{C}^{1,16} \xrightarrow{d} \mathcal{F}_2\mathcal{C}^{2,16}$  is given below. Here and often we omit  $\varepsilon_i$  from the notation. It is always implicit; we are always working with  $S^{0\langle i \rangle}$ , although in this case  $i=2$ .

	$w^2 \tau_1 \tau_1$	$w^1 \tau_1 \tau_2$	$w^1 \tau_2 \tau_1$	$\tau_1 \tau_3$	$\tau_2 \tau_2$	$\tau_3 \tau_1$
$8w^3 \tau_1$	$8\binom{3}{1}$		$8\binom{3}{2}$			8
$w^2 \tau_2$	$-\binom{2}{1}$	$\binom{2}{1}$			1	
$w^1 \tau_3$		$-\binom{3}{1}$	$-\binom{3}{2}$	1		
$\tau_4$				$-\binom{4}{3}$	$-\binom{4}{2}$	$-\binom{4}{1}$

Because the total complex  $\mathcal{F}_2\mathcal{C}^*$ , modified so that there are no restrictions on coefficients, is acyclic,  $\ker(d)$  consists precisely of multiples (perhaps by fractions with 2 in the denominator) of elements of  $\text{im}(d)$ .

Row reducing the matrix, we obtain the following matrix, where  $u$  and  $v$  are certain odd integers. The names of the columns and second and third rows are the same as above. The fourth row is  $\tau_4 + \binom{4}{3}w^1 \tau_3 + \binom{4}{2}w^2 \tau_2$ , and the new first row is  $8w^3 \tau_1 + 2 \cdot \text{odd new 4th}$ .

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\binom{2}{1} & \binom{2}{1} & & & 1 & \\ & -\binom{3}{1} & -\binom{3}{2} & 1 & & \\ 4uv & & 4uv & & 4u & \end{pmatrix}$$

Thus  $\frac{1}{4}d(\tau_4 + \binom{4}{3}w^1 \tau_3 + \binom{4}{2}w^2 \tau_2) = uvw^2 \tau_1 \tau_1 + uvw^1 \tau_2 \tau_1 + u\tau_3 \tau_1$  generates  $H^{2,16}(\mathcal{F}_2\mathcal{C}) \approx \mathbb{Z}/4$ .

The weight-preserving ( $\partial_0$ ) portion of the weight spectral sequence (WSS) of [22, §5,6] goes as claimed there, yielding as  $E_1^{*,*,*}$  for  $s \geq 2$  a  $\mathbb{F}_2$ -vector space for which a useful set of representatives consists of

$$\{\frac{1}{2}D_0 w^m h^I : I=(i_0, \dots, i_{r-1}, 1), 2'(m - \|I\|) \leq i - s\}.$$

Here, as in [22], if  $I=(i_0, \dots, i_r)$ , then  $h^I$  denotes the product of  $i_0 \tau_2$ 's followed by  $i_1 \tau_2$ 's, etc. Note that  $i$  in the inequality refers to the  $i$  in  $\mathcal{F}_i$ , not in  $I$ . The element  $\frac{1}{2}D_0 w^m h^I$  contributes to  $E_1^{s,t}$  with weight  $\sigma = \|I\| = \sum 2^j i_j$ ,  $s = 1 + \sum i_j$ , and  $t = 4(\sigma + m)$ . As in [22,

p. 612], the pattern of differentials is

$$(5.4) \quad \partial_{2^e}(\tfrac{1}{2} D_0 w^m h^I) = \tfrac{1}{2} D_0 w^{m-2^e} h_e h^I$$

whenever  $v(m)=e$  and  $I$  begins with at least  $e$  0s.

There is a problem here when  $h^I = h_e$ , for then the RHS of (5.4) has  $(0, \dots, 0, 2)$  as its (vector) exponent of  $h$ , and this is not of the requisite form. This allows the set

$$\{\tfrac{1}{2} D_0 w^{m-2^e} h_e : 1 \leq e \leq v(m), 2'(m-2^{e+1}) \leq i-2\}$$

to survive these differentials in  $s=2$ . The ones with  $v(m)=e$  were the elements obtained in [22, 6.6], while those with  $e < v(m)$  are new. Note that this problem does not occur if  $m$  is odd, i.e. it only occurs in total degree  $\equiv 0 \pmod 8$ . From now on, we will assume  $m$  is even. The  $\tfrac{1}{2} D_0(\ )$  notation was useful in [22] in showing that  $2H^{*,*} \mathcal{F}_i \mathcal{C} = 0$ ; however, replacing  $\tfrac{1}{2} D_0 w^{m-2^e} h_e$  by its only term  $w^{m-2^e} h_{e-1}^2$  with minimal weight and odd coefficient may simplify the notation.

Some of these new classes will be killed by differentials in the WSS from elements with  $s=1$ . To see this, we recall

$$E_1^{\sigma, 0, 4m} = \begin{cases} \mathbf{Z}_{(2)} & \sigma = 0 \\ 0 & \sigma \neq 0 \end{cases}$$

with generator  $2^b w^m$ , where  $b = \max(0, 2'm - i)$ , and

$$E_1^{\sigma, 1, 4m} = \begin{cases} \mathbf{Z}_{(2)} & \sigma = 1, m > 0 \\ 0 & \sigma \neq 1 \end{cases}$$

with generator  $2^a w^{m-1}[\tau_1]$  with

$$a = \max(0, 2'm - 3 - i).$$

The obvious  $\partial_1$ -differential yields

$$E_2^{\sigma, 0, 4m} = \begin{cases} \mathbf{Z}_{(2)} & \sigma = 0, m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

while, if  $m \geq 1$ ,  $E_2^{1, 1, 4m}$  is cyclic of order  $2^a$  generated by  $2^a w^{m-1}[\tau_1]$ , where

$$q = \begin{cases} 3 + v(m) & \text{if } i \leq 2'm - 3 \\ 2'm - i + v(m) & \text{if } 2'm - 3 < i \leq 2'm \\ v(m) & \text{if } 2'm < i. \end{cases}$$

In order to help the reader see the next round of differentials and that the  $\mathbf{Z}/2$ s which survive the WSS in  $s=2$  build up to a cyclic group in  $H^{2, 4m}(\mathcal{F}_i \mathcal{C})$ , we consider the case  $m=32, i=57$ . Here  $E_2^{1, 1, 128} \approx \mathbf{Z}/2^8$  generated by  $2^4 w^{31} h_0$ . We try to extend this to a cycle in the actual complex  $\mathcal{F}_i \mathcal{C}$ . Ignoring odd factors in the coefficients for simplicity (but see 5.5, where they are included),

$$\tfrac{1}{2} d(w^{32}) = \sum_{i=1}^{32} 2^{4-v(i)} w^{32-i}[\tau_i]$$

is certainly a cycle and has correct term  $2^4 w^{31}[\tau_1]$  of minimal weight, but its final term fails to be a legitimate element of  $\mathcal{F}_i \mathcal{C}$ . We use the sum

$$A = \sum_{i=1}^{31} 2^{4-v(i)} w^{32-i}[\tau_i]$$

to represent the term in  $E_1$ .

Now  $d(A) = -d(\frac{1}{2}[\tau_{32}]) \equiv h_4^2 \pmod{2}$ . Next we note

$$E_2^{\sigma, 2, 128} = \begin{cases} \mathbf{Z}/2 & \sigma = 4, 8, 16, 32 \\ 0 & \text{otherwise} \end{cases}$$

with generator  $w^{32-2^e}h_{e-1}^2$  if  $\sigma = 2^e$ . Because of the analysis of  $d(A)$ , there is a nonzero differential

$$\begin{array}{ccc} E_{31}^{1, 1, 128} & \xrightarrow{\hat{c}_{31}} & E_{31}^{32, 2, 128} \\ \mathbf{Z}/2^8 & & \mathbf{Z}/2. \end{array}$$

An easier way to see this differential and to see that  $H^{2, 128}(\mathcal{F}_{57}\mathcal{C}) \approx \mathbf{Z}/8$  (which we will later show from the WSS point of view) is to consider the quotient complex

$$Q^{*,*} = C^{*,*}/\mathcal{F}_i\mathcal{C}^{*,*},$$

where  $C^{*,*}$  is the acyclic complex with basis like that of (5.3) with  $a=0$  for all monomials. The exact homology sequence associated to

$$0 \rightarrow \mathcal{F}_i\mathcal{C} \rightarrow C \rightarrow Q \rightarrow 0$$

yields an isomorphism

$$H^{s, 4m}(Q) \xrightarrow{\delta} H^{s+1, 4m}(\mathcal{F}_i\mathcal{C}).$$

In our example case  $i=57$ ,  $m=32$ , the entire complex  $Q^{*, 128}$  is

$$\begin{array}{ccccc} 0 & & 1 & & 2 \\ \mathbf{Z}/2^7 & \xrightarrow{\langle \begin{smallmatrix} 32 \\ 1 \end{smallmatrix} \rangle} & \mathbf{Z}/2^4 & \xrightarrow{\langle \begin{smallmatrix} 31 \\ 1 \end{smallmatrix} \rangle} & \mathbf{Z}/2 \\ w^{32} & & w^{31}[\tau_1] & & w^{30}[\tau_1|\tau_1], \end{array}$$

from which we clearly have

$$H^{s, 128}(\mathcal{F}_{57}\mathcal{C}) \approx \begin{cases} \mathbf{Z}/2^7 & s=1 \\ \mathbf{Z}/2^3 & s=2 \\ 0 & \text{otherwise} \end{cases}$$

with generators  $d(w^{32})$  and  $d(2w^{31}\tau_1)$ . Unfortunately, this quotient complex method becomes more complicated as  $2m-i$  becomes large.

The elements  $w^{28}h_1^2$ ,  $w^{24}h_2^2$ , and  $w^{16}h_3^2$ , which survive the WSS to  $E_{\infty}^{*, 2, 128}$ , build up a  $\mathbf{Z}/8$  because  $2w^{28}h_1^2 - w^{24}h_2^2$  form the lead terms of the boundary  $d(w^{28}h_2)$ , and similarly  $2w^{24}h_2^2$  differs from  $w^{16}h_3^2$  by a boundary.

In the general case, we have the following result.

**PROPOSITION 5.5.** *Suppose  $m$  is even and  $i \geq 2$ . Let  $l(N) = \lceil \log_2(N) \rceil$ . Then the set of nonzero elements of  $E_{\infty}^{*, 2, 4m}(\mathcal{F}_i\mathcal{C})$  is*

$$S = \{w^{m-2^e}h_{e-1}^2 : \max(1, l(2m-i+1)-1) \leq e \leq \min(2m-i-3, v(m))\},$$

and  $H^{2, 4m}(\mathcal{F}_i\mathcal{C}) \approx \mathbf{Z}/2^{|S|}$ . A generator is

$$\frac{1}{2\langle \begin{smallmatrix} m \\ 2^e \end{smallmatrix} \rangle} d\left(\sum_{j \geq 2^e} \langle \begin{smallmatrix} m \\ j \end{smallmatrix} \rangle w^{m-j}[\tau_j]\right),$$

where  $e = l(2m-i+1)-1$ .

*Proof.* The pattern of differentials which leaves the elements  $S$  is similar to that illustrated in the example above. Let  $v = v(m)$ . If  $i \geq 2m - 3$ , then  $E_2^{1, \cdot, 4m}$  is generated by  $w^{m-1}h_0$  and there are differentials (again omitting the writing of odd factors in coefficients)

$$\begin{aligned} w^{m-1}h_0 &\rightarrow w^{m-2}h_0^2 \\ 2w^{m-1}[\tau_1] + w^{m-2}[\tau_2] &\rightarrow w^{m-4}h_1^2 \\ &\vdots \\ 2^{v-1}w^{m-1}[\tau_1] + \cdots + w^{m-2^{v-1}+1}[\tau_{2^{v-1}-1}] &\rightarrow w^{m-2^v}h_{v-1}^2. \end{aligned}$$

If  $i = 2m - 3 - b$  with  $1 \leq b \leq v$ , then there are differentials as above beginning with the one hitting  $w^{m-2^{b+1}}h_b^2$ .

The term of lowest weight and odd coefficient in the generator proposed in the proposition is the element of  $S$  of minimal  $e$ . To see that this is a legitimate element of  $\mathcal{F}_i\mathcal{C}$ , we observe that, since  $d \circ d = 0$ , it equals the negative of a similar expression summed over  $j < 2^e$ , and in this sum all terms of weight  $\geq 2^e$  are legitimate, i.e. multiples of elements of (5.3). The multiples of the generator of 5.5 by increasing 2-powers represent the various elements of  $S$ .  $\blacksquare$

These differentials cause a slight change in  $H^{1, 4m}(\mathcal{F}_i\mathcal{C})$  when  $i \geq 2$ , which we shall not write out.

This change in [22, 6.6] causes a corresponding modification of [22, 7.2], the proof being similar to that given there.

PROPOSITION 5.6. *If  $\iota: S^{0\langle i \rangle} \rightarrow S^{0\langle i-1 \rangle}$  denotes the canonical map, then*

$$\iota_*^{s, t}: H^{s, t}(\mathcal{F}_i\mathcal{C}) \rightarrow H^{s, t}(\mathcal{F}_{i-1}\mathcal{C})$$

*is zero if  $i \geq 1$  and  $s > 2$ . If  $s = 2$ , then  $\iota_*^{s, 4m}$  is surjective if  $2m + 2 - i$  is a 2-power, and has cokernel  $\mathbf{Z}/2$  otherwise.*

This has the following immediate corollary.

COROLLARY 5.7. *The homomorphism*

$$H^{2, 4m} \mathcal{F}_{i+v(m)}\mathcal{C} \rightarrow H^{2, 4m} \mathcal{F}_i\mathcal{C}$$

*induced by  $\iota^{v(m)}: S^{0\langle i+v(m) \rangle} \rightarrow S^{0\langle i \rangle}$  is 0.*

This could be refined greatly; the maximal change of subscript  $v(m)$  is required only for  $2m - v(m) - 6 \leq i \leq 2m - v(m) - 4$ . For example, if  $v(m) = 5$ , we tabulate below the smallest and largest values of  $e$  appearing in  $S$  of 5.5. In  $H^{2, 4m}(\mathcal{F}_i\mathcal{C})$  these form a cyclic group with  $2 \cdot e$ -class =  $(e + 1)$ -class. The morphism of 5.6 sends  $e$ -class to  $(e + 1)$ -class.

$2m - i$	4	5	6	7	8-14	15-30	31-62	63-126
bottom $e$	0	0	0	1	1	2	3	4
top $e$	0	1	2	3	4	4	4	4

The last column does not appear if  $m = 32$ .

There is an analogous change in [22, 7.1], for which we will let our 5.1 serve as a substitute. We have not analyzed the extent to which the results of [22, 6.6, 7.1] for stunted projective spaces are affected by the difficulties which we have been discussing.

In order to prove 5.1, we will need the following lemma.



LEMMA 5.8. Suppose  $X \xrightarrow{f} Y \xrightarrow{g} Z$  is a cofibration of spectra, and  $Z$  is a wedge of suspensions of  $HF_2$ ,  $bo^{(i)}$ , and  $bsp^{(i)}$ . For example,  $Z = \overline{bo}^s \wedge bo$ . If  $\alpha \in \pi_n Y$  satisfies  $AF(\alpha) = a$  and  $g_* \alpha = 0$ , then there exists  $\beta \in \pi_n X$  with  $AF(\beta) \geq a-1$  and  $f_* \beta = \alpha$ .

*Proof.* Let  $\rightarrow A_i \rightarrow \cdots \rightarrow A_1 \rightarrow S^0$  denote an  $(HF_2)$ -Adams resolution of  $S^0$ . Then there is a commutative diagram of Adams resolutions with horizontal cofibrations

$$\begin{array}{ccccc}
 X \wedge A_a & \longrightarrow & Y \wedge A_a & \xrightarrow{g_a} & Z \wedge A_a \\
 \downarrow & & \downarrow p_Y & & \downarrow p_Z \\
 X \wedge A_{a-1} & \longrightarrow & Y \wedge A_{a-1} & \longrightarrow & Z \wedge A_{a-1} \\
 \downarrow & \xrightarrow{f_{a-1}} & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow \\
 X \wedge A_1 & \longrightarrow & Y \wedge A_1 & \longrightarrow & Z \wedge A_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & Y & \longrightarrow & Z.
 \end{array}$$

Let  $\alpha_a \in \pi_n(Y \wedge A_a)$  be a lifting of  $\alpha$ . We will show that  $p_{Z*} g_{a*}(\alpha_a) = 0$ . Thus there is  $\beta_{a-1} \in \pi_n(X \wedge A_{a-1})$  such that  $f_{a-1*}(\beta_{a-1}) = p_{Y*}(\alpha_a)$ . The image of  $\beta_{a-1}$  in  $\pi_n X$  works as  $\beta$ .

To prove the claim of the above paragraph, we note that for all  $j$

$$\pi_n(Z \wedge A_j) \rightarrow \pi_n(Z \wedge A_{j-1})$$

is a direct sum of injections of infinite cyclic groups and zero homomorphisms of  $\mathbb{Z}/2$ s. Our class  $g_{a*}(\alpha_a) \in \pi_n(Z \wedge A_a)$  goes to 0 in  $\pi_n(Z)$ , and so must be 0 or one of the  $\mathbb{Z}/2$ s which goes to 0 in  $\pi_n(Z \wedge A_{a-1})$ .  $\blacksquare$

Now we prove 5.1, which, although technical, is the most central new ingredient of this paper. The easiest case is (i) when  $n+s \equiv 3, 5, 6, 7 \pmod{8}$ . In this case  $\pi_n(S_s \wedge bo) \approx \pi_{n+s}(\overline{bo}^s \wedge bo)$  is 0 above filtration 0, since  $\pi_{n+s+8j}(bo) = 0$ . Thus in 5.1,  $i_s \circ f_s = 0$ , and the conclusion follows by 5.8 applied to  $S_{s+1} \rightarrow S_s \rightarrow S_s \wedge bo$ .

In the other case of 5.1(i), the class  $[i_s \circ f_s] \in \pi_n(S_s \wedge bo)$  is a  $d_1$ -cycle in the  $bo$ -ASS, and so by [22, 7.2] (since  $s \geq 3$ ) there is an element  $\bar{h} \in \mathcal{C}^{s-1, n+s}$  so that  $\bar{d}_1(\bar{h}) = \overline{i_s \circ f_s}$  and  $AF(\bar{h}) \geq a-1$ . The bars refer to the commutative diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \pi_n(\vee HF_2) & \longrightarrow & \pi_n(\vee HF_2) \\
 \downarrow & & \downarrow \\
 \pi_n(\Sigma^{-1} S_{s-1} \wedge bo) & \xrightarrow{d_1} & \pi_n(S_s \wedge bo) \\
 \downarrow - & & \downarrow - \\
 \mathcal{C}^{s-1, n+s} & \xrightarrow{\bar{d}_1} & \mathcal{C}^{s, n+s} \\
 \downarrow & & \downarrow \\
 0 & & 0.
 \end{array}$$

Since the images from  $\pi_*(VHF_2)$  have filtration 0, and  $a \geq 2$ , the diagram implies there is  $h \in \pi_n(\Sigma^{-1}S_{s-1} \wedge bo)$  such that  $d_1(h) = [\iota_s \circ f_s]$  and  $AF(h) \geq a-1$ . Recalling the diagram

$$\begin{array}{ccccc} & & S_{s+1} & & \\ & & \downarrow p_s & & \\ \Sigma^{-1}S_{s-1} \wedge bo & \xrightarrow{j_s} & S_s & \xrightarrow{\iota_s} & S_s \wedge bo, \\ & \nearrow f_s & & & \\ & S^n & & & \end{array}$$

$\iota_s(f_s - j_s \circ h) = 0$  and  $AF(f_s - j_s \circ h) \geq a-1$ , so that 5.8 yields the desired result.

Part (ii) follows similarly. Let  $v = v(n+2)$ . By 5.7, the element  $\overline{\iota_s \circ f_s} \in H^{2, n+2}(\mathcal{F}_a \mathcal{C})$  goes to 0 in  $H^{2, n+2}(\mathcal{F}_{a-v} \mathcal{C})$ . Thus there is  $\bar{h} \in \mathcal{C}^{1, n+2}$  with  $\bar{d}_1(\bar{h}) = \overline{\iota_s \circ f_s}$  and  $AF(\bar{h}) \geq a-v > 0$ . Here we have used injectivity of  $\mathcal{F}_{a-v} \mathcal{C} \rightarrow \mathcal{C}$ . As before,  $\bar{h}$  pulls back to an element  $h \in \pi_n(\Sigma^{-1}S_1 \wedge bo)$  with  $AF(h) \geq a-v$  and  $d_1(h) = \iota_s \circ f_s$ . The map  $f_{s+1}$  is obtained as before as a lifting of  $f_s - j_s \circ h$ .

The case  $s=0$  in (iii) is elementary because of the factorization  $S^0 \rightarrow J \rightarrow bo$ . For  $s=1$  we use the commutative diagram

$$\begin{array}{ccccc} S_1 & \xrightarrow{\iota_1} & S_1 \wedge bo & \xrightarrow{s} & \Sigma^3 bsp \\ \downarrow p_1 & & & & \downarrow \\ S^0 & \xrightarrow{\quad} & J & & \downarrow \\ & & & & bo \end{array}$$

By [22, 5.1] and [14, 3.2] the only  $d_1$ -cycles in  $\pi_n(S_1 \wedge bo)$  are the infinite cyclic summand which maps onto  $\pi_n(J)$  when  $n \equiv 3 \pmod{4}$  and the  $\mathbb{Z}/2$  for  $n \equiv 0, 1 \pmod{8}$ . Since  $[\iota_s \circ f_s]$  is a  $d_1$ -cycle and  $i_{j_*}(f_s) = 0$ , we must have  $n \equiv 3 \pmod{4}$  and  $[\iota_s \circ f_s]$  is hit by a  $d_1$ -differential, and this increases Adams filtration by  $v(n+1)-2$ . The rest of the argument follows as before.

## 6. PROOF OF THEOREM 1.2

Theorem 1.2 is an easy consequence of 5.1 and the following result, whose proof was sketched in [28] and given in detail in [14].

**THEOREM 6.1.** *In the bo-ASS converging to  $\pi_*(S^0)$ ,*

$$E_2^{s,t} = 0 \quad \text{if } t-s < \begin{cases} 3s & s \leq 6 \\ 5s-13 & s > 6. \end{cases}$$

Let  $\alpha$  be as in 1.2 with  $a = AF(\alpha)$ . We use the notation for the bo-resolution as in the beginning of Section 5. By 5.1(iii) there exists a lifting to an element  $\alpha_2 \in \pi_i(S_2)$  with  $AF(\alpha_2) \geq a - \max(2, v(i+1))$ . Now 5.1(ii) assures a lifting to  $\alpha_3 \in \pi_i(S_3)$  with  $AF(\alpha_3) \geq a - M$ , where

$$M = \max(v(i+2)+3, v(i+2)+v(i+1)+1).$$

One may continue lifting up the bo-resolution by 5.1(i), losing no more than 12 Adams filtrations for every 8 stages lifted, until the Adams filtration drops below 2. Thus a lifting to

$S_{s_0}$  with  $s_0 = \frac{2}{3}(a - M + \frac{7}{2})$  is obtained. If  $i < \max(3s_0, 5s_0 - 13)$ , then by 6.1  $\alpha = 0$ , establishing 1.2, in which we have weakened the inequalities to make them less complex.

## REFERENCES

1. J. F. ADAMS: *Infinite Loop Spaces*, Princeton Univ. Press, 1978.
2. J. F. ADAMS: On the groups  $J(X)$ -I-IV, *Topology* **2,3,3,5** (1963, 5, 5, 6), 181-195, 137-171, 193-222, 21-71.
3. J. F. ADAMS: A periodicity theorem in homological algebra, *Proc. Camb. Phil. Soc.* **62** (1966), 365-377.
4. J. F. ADAMS: Operations of the  $n$ th kind in  $K$ -theory and what we don't know about  $\mathbb{R}P^\infty$ , *London Math. Soc. Lecture Notes* **11** (1974), 1-9.
5. J. F. ADAMS: The Kahn-Priddy theorem, *Math. Proc. Camb. Phil. Soc.* **73** (1973), 45-55.
6. D. W. ANDERSON: The  $e$ -invariant and the Hopf invariant, *Topology* **9** (1970), 49-54.
7. M. G. BARRATT and M. MAHOWALD: The metastable homotopy of  $O(n)$ , *Bull. Amer. Math. Soc.* **70** (1964), 758-760.
8. J. CARUSO, F. R. COHEN, J. P. MAY, and L. R. TAYLOR: James maps, Segal maps, and the Kahn-Priddy theorem, *Trans. Amer. Math. Soc.* **281** (1984), 243-283.
9. M. C. CRABB: Desuspension of the image of  $J$ , preprint.
10. M. C. CRABB and K. KNAPP: Adams periodicity in stable homotopy, *Topology* **24** (1985), 475-486.
11. E. B. CURTIS: Simplicial homotopy theory, *Adv. in Math.* **6** (1971), 107-209.
12. E. B. CURTIS: Some nonzero homotopy groups of spheres, *Bull. Amer. Math. Soc.* **75** (1969), 541-544.
13. D. M. DAVIS: Generalized homology and the generalized vector field problem, *Quar. Jour. Math.* **25** (1974), 169-193.
14. D. M. DAVIS: The  $bo$ -Adams spectral sequence: some calculations and a proof of its vanishing line, Springer Verlag Lecture Notes in Math, **1286** (1987), 267-285.
15. D. M. DAVIS, S. GITLER and M. MAHOWALD: The stable geometric dimension of vector bundles over real projective spaces, *Trans. Amer. Math. Soc.* **268** (1981), 39-62.
16. D. M. DAVIS and M. MAHOWALD: Homotopy groups of some mapping telescopes, Algebraic Topology and Algebraic  $K$ -Theory, *Ann. Math. Stud.* **113** (1987), 126-151.
17. D. M. DAVIS and M. MAHOWALD:  $v_1$  and  $v_2$ -periodicity in stable homotopy theory, *Amer. J. Math.* **103** (1981), 615-659.
18. D. M. DAVIS and M. MAHOWALD: The  $SO(n)$ -of-origin, to appear.
19. S. FEDER, S. GITLER and K. Y. LAM: Composition properties of projective homotopy classes, *Pac. J. Math.* **68** (1977), 47-61.
20. D. S. KAHN and S. B. PRIDDY: Applications of the transfer to stable homotopy theory, *Bull. Amer. Math. Soc.* **76** (1972), 981-987.
21. N. J. KUHN: The geometry of the James-Hopf maps, *Pac. J. Math.* **102** (1982), 397-412.
22. W. LELLMAN and M. MAHOWALD: The  $bo$ -Adams spectral sequence, *Trans. Am. Math. Soc.* **300** (1987), 593-623.
23. J. M. LONG: Two contributions to the homotopy theory of  $H$ -spaces, Thesis, Princeton Univ. (1979).
24. M. MAHOWALD: The order of the image of the  $J$ -homomorphism, *Bull. Am. Math. Soc.* **76** (1970), 1310-1313.
25. M. MAHOWALD: Descriptive homotopy of the image of  $J$ , *Proc. of Int. Conf. on manifolds* (1975), 245-253, Univ. of Tokyo Press.
26. M. MAHOWALD: The order of the image of the  $J$ -homomorphism, *Proc. Adv. Study Inst. Aarhus* (1970), 376-384.
27. M. MAHOWALD: The image of  $J$  in the  $EHP$  sequence, *Ann. Math.* **116** (1982), 65-112.
28. M. MAHOWALD:  $bo$ -resolutions, *Pac. J. Math.* **192** (1981), 365-383.
29. M. MAHOWALD: On the order of the image of  $J$ , *Topology* **6** (1967), 371-378.
30. M. MAHOWALD: The metastable homotopy of  $S^n$ , *Mem. Amer. Math. Soc.* **72** (1967).
31. M. MAHOWALD and R. J. MILGRAM: Operations that detect  $Sq^4$  in connective  $K$ -theory and their applications, *Quar. Jour. Math.* **27** (1976), 415-432.
32. H. R. MARGOLIS, S. B. PRIDDY and M. C. TANGORA: Another systematic phenomenon in the cohomology of the Steenrod algebra, *Topology* **10** (1971), 43-46.
33. J. P. MAY: The geometry of iterated loop spaces, Springer Verlag Lecture Notes in Math **271** (1972).
34. D. G. QUILLLEN: The Adams conjecture, *Topology* **10** (1971), 1-10.
35. G. SEGAL: Operations in stable homotopy, *London Math Soc Lecture Notes* **11** (1974), 105-110.
36. P. SELICK: 2-primary exponents for the stable homotopy groups of spheres, *Topology* **23** (1984), 97-99.
37. V. P. SNAITH: A stable decomposition for  $\Omega^n \Sigma^n X$ , *J. Lond. Math. Soc.* **7** (1974), 577-583.
38. R. E. STONG: Determination of  $H^*(BO(k, \dots); \mathbb{Z}_2)$  and  $H^*(BU(k, \dots); \mathbb{Z}_2)$ , *Trans. Am. Math. Soc.* **107** (1963), 526-544.

- 39. D. P. SULLIVAN: Genetics of homotopy theory and the Adams conjecture, *Ann. Math.* **100** (1974), 1–79.
- 40. M. C. TANGORA: On the cohomology of the Steenrod algebra, *Math. Zeit.* **116** (1970), 18–64.
- 41. G. W. WHITEHEAD: On the homotopy groups of spheres and rotation groups, *Ann. Math.* **43** (1942), 634–640.

*Lehigh University,*  
*Bethlehem, PA 18015*  
*U.S.A.*

*Northwestern University,*  
*Evanston, IL 60201*  
*U.S.A.*